Algebraic Geometry Lecture 18 – Categories, schemes, and sheaves

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1. Natural equivalence

Given two functors $S, T : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : S \to T$ is a function assigning to each $c \in \mathrm{ob}(\mathcal{C})$ an arrow in $\mathcal{D}$, $\tau_c : S(c) \to T(c)$, such that for every arrow $f : c \to c'$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccccc} c & \quad & S(c) & \xrightarrow{\tau_c} & T(c) \\ {\scriptstyle f}\downarrow & & {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ c' & & S(c') & \xrightarrow{\tau_{c'}} & T(c') \end{array}$$

If every component τ_c of τ is invertible in $\mathcal{D}$ then we say τ is a natural equivalence of functors (or a natural isomorphism). We write $\tau : \mathcal{C} \simeq \mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that there exist natural transformations $\tau : GF \simeq \mathrm{id}_{\mathcal{C}}$ and $\sigma : FG \simeq \mathrm{id}_{\mathcal{D}}$. (If $GF \simeq \mathrm{id}_{\mathcal{C}}$ and $FG \simeq \mathrm{id}_{\mathcal{D}}$ then we say F is isomorphic to G, but this isn't actually that useful.)

E.g. 1. Take $\mathcal{C}$ to be the category of finite sets and take $\mathcal{D} = \mathbb{N}$. $\mathcal{C}$ and $\mathcal{D}$ are naturally equivalent categories.

Opposite categories.

Given a category $\mathcal{C}$ we can define the opposite category $\mathcal{C}^{\mathrm{op}}$ by:

- $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$,
- $\mathrm{mor}(\mathcal{C}^{\mathrm{op}})$ is defined by:

$$\exists\ f^{\mathrm{op}} : c \to c' \text{ in } \mathcal{C}^{\mathrm{op}} \iff \exists\ f : c' \to c \text{ in } \mathcal{C}.$$

$\mathcal{C}$	$\Leftrightarrow$	$\mathcal{C}^{\mathrm{op}}$
$f : a \to b$		$f^{\mathrm{op}} : b^{\mathrm{op}} \to a^{\mathrm{op}}$
$h = g \circ f$		$h^{\mathrm{op}} = f^{\mathrm{op}} \circ g^{\mathrm{op}}$
f is monic		f^{op} is epi
$i = \mathrm{id}_a$		$i^{\mathrm{op}} = \mathrm{id}_{a^{\mathrm{op}}}$
f is invertible		f^{op} is invertible.

[1]Notes typed by Lee Butler based on a lecture given by Joe Grant. Any errors are the responsibility of the typist. Or aliens.

2. Algebraic geometry

Let

$$\begin{aligned} \mathcal{C} &:= \text{objects are algebraic sets in } \mathbb{A}^n \text{ (over } k) \\ &\quad\ \text{morphisms exist from } S \to T \text{ if and only if } S \subseteq T. \end{aligned}$$

$$\begin{aligned} \mathcal{D} &:= \text{objects are radical ideals in } k[X_1, \dots, X_n] \\ &\quad\ \text{morphisms exist from } I \to J \text{ if and only if } I \subseteq J. \end{aligned}$$

Corollary of Nullstellensatz.

There is a natural equivalence $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$.

A lot of important ideas in algebraic geometry have the form $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$.

Affine schemes.

Given a commutative ring with identity, R, an affine scheme is three things:

(1) A set of points: $X = \operatorname{spec} R := \{\text{ prime ideals of } R\}$.

(2) A topology on X: Zariski topology, closed sets are $V(S)$ for $S \subseteq R$ where $V(S) = \{P \in X \mid S \subseteq P\}$. (In a previous lecture, Andrew showed us how to think of (S) as a function on $\operatorname{spec} R$. Using this correspondence, $V(S) = \{x \in \operatorname{spec} R \mid f(x) = 0 \ \forall\ f \in S\}$.) The open sets are the complements of the closed sets. Dan showed that this forms a topology.

Sets $S \subseteq R$ where $S = \{f\}$ for $f \in R$ are easy to understand. Define the distinguished (basic) open subset of $X = \operatorname{spec} R$ associated to f to be

$$X_f = \operatorname{spec} R \smallsetminus V(f).$$

Remember for a ring R and $f \in R$ we write R_f to denote the localisation of R with respect to f, obtained by adjoining formal inverses. (E.g. if $R = \mathbb{Z}$ and $f = 2$ then $R_f = \{\frac{a}{2^b} \mid a, b \in \mathbb{Z}\}$.)

We then obtain a $1-1$ correspondence

$$\begin{array}{ccc} \{\text{ points of } X_f\} & \overset{1-1}{\leftrightarrow} & \{\text{prime ideals of } R_f\} \\ P & \mapsto & PR_f \\ \hookrightarrow & & \hookrightarrow \\ R & & R_f \end{array}$$

Exercise: check this is $1-1$

The open sets X_f, $f \in R$, form a base for the topology on X; for an arbitrary open set, U, we have

$$\begin{aligned} U &= \operatorname{spec} R \smallsetminus V(S) \qquad S \in R \\ &= \operatorname{spec} R \smallsetminus \bigcap_{f \in S} V(f) \\ &= \bigcup_{f \in S} (\operatorname{spec} R)_f. \end{aligned}$$

(3) A structure sheaf/sheaf of regular functions.

Recall: Let X be a topological space, then define the category

$$\underline{\text{top}}(X) = \begin{cases} \text{Objects are open sets of } X \\ \text{There is a morphism } f : U \to V \text{ iff } U \subseteq V. \end{cases}$$

Then a presheaf is just a functor $F : \underline{\text{top}}(X) \to \text{Set}^{\text{op}}$ (or $F : \underline{\text{top}}(X) \to \underline{\text{Comm.Ring}}^{\text{op}}$). Then if we define a functor

$$\mathcal{O}_X : \underline{\text{top}}(B) \to \underline{\text{Comm.Ring}}^{\text{op}}$$

where B is the base set of X, by

$$\mathcal{O}_X(\operatorname{spec} R_f) = R_f$$

this can be extended uniquely to a presheaf

$$\mathcal{O}_X : \underline{\text{top}}(X) \to \underline{\text{Comm.Ring}}^{\text{op}}$$

which can be sheafified.